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WHAT MUST AND WHAT NEED NOT BE CONTAINED IN A GRAPH OF UNCOUNTABLE CHROMATIC NUMBER?

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Dedicated to Paul Erdős on his seventieth birthday

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We investigate the following problem: What countable graphs must a graph of uncountable chromatic number contain? We define two graphs Γ and Δ which are very similar and we show that Γ is contained in every graph of uncountable chromatic number, while Δ is (consistently) not.

0. Introduction

In this paper graphs with uncountable chromatic numbers will be studied. As usual, a *graph* is an ordered pair $G=\langle V, E\rangle$, where V is an arbitrary set (the set of *vertices*), E is a set of unordered pairs from V (the set of *edges*). A function $: V\to\varkappa$ ($\varkappa$ a cardinal). is a *good coloring* of G if and only if $f(x)\neq f(y)$ whenever x and y are joined; i.e. joined vertices get different colors. The *chromatic number* of G, Chr (G) is the minimal cardinal $\varkappa$ onto which a good coloring of G exists.

The following statement was proved by Tutte, Zykov, Ungár, Mycielski and possibly by many other people (see e.g. [9], [11]): for every finite n there exists an n-chromatic triangle-free graph. P. Erdős and R. Rado proved in [5] that for every cardinality $\varkappa$ there exists a triangle-free $\varkappa$-chromatic graph of cardinality $\varkappa$. P. Erdős proved (see [1]) that for any given finite n and s there exist n-chromatic graphs without circuits of length $\leqq s$. His proof was non-constructive. A constructive proof was later found by L. Lovász [8]. However quite surprisingly, the natural common generalization of these theorems turned out to be false: if a graph has chromatic number $\geqq\aleph_1$ then it necessarily contains a four-cycle, moreover every finite bipartite graph must be contained in such a graph. As examples show, for every $s<\omega$ and arbitrary cardinality $\varkappa$ there exist graphs with chromatic number $\varkappa$ (and of cardinality $\varkappa$) without any odd circuit of length $\leqq s$ (see [3]). Thus, all finite obligatory graphs are described. The next problems are the following: find the obligatory classes of finite graphs and the obligatory countable graphs. Both problems seem to be very difficult (see [4]). In this paper we try to attack the second problem by displaying two graphs Γ and Δ which are very close together and though Γ is obligatory Δ is not (at least consistently).

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First we show that if $\mathrm{Chr}(G) \geqq \aleph_1$ then G contains the countable semicomplete bipartite graph, even the following graph Γ: $V(\Gamma)$ consists of the different points $\{x_i, y_i, a: i<\omega\}$ and y_i is joined to $\{x_0, \ldots, x_{i-1}\}$ for $i<\omega$, a is joined to $\{x_i: i<\omega\}$.

With the use of the continuum hypothesis (CH) we give an example of a graph with chromatic number $\aleph_1$ not containing the following graph Δ: $V(\Delta)$ consists of the different points $\{x_i, y_i, a, b: i<\omega\}$ and y_i is joined to $\{x_0, \ldots, x_{i-1}\}$ for $i<\omega$, a and b are joined to $\{x_i: i<\omega\}$. For this, we use a combination of methods described in [2] and [7].

Using this construction we answer a problem of [4]. For $2 \leqq k < \omega$ let $G_k(\lambda)$ denote the k-shift graph on λ, i.e.

$$V(G_k(\lambda)) = [\lambda]^k,$$

$$E(G_k(\lambda)) = \big\{\{\{x_0, \ldots, x_{k-1}\}_<, \{x_1, \ldots, x_k\}_<\}: \{x_0, \ldots, x_k\}_< \in [\lambda]^{k+1}\big\}.$$

Let $\mathscr{S}(G)$ denote the set of all finite subgraphs contained in G. In [4] it was asked if $\mathrm{Chr}(G) \geqq \aleph_1$ implies that

$$\mathscr{S}(G_k(\omega)) \subset \mathscr{S}(G) \quad \text{for some} \quad 2 \leqq k < \omega.$$

We prove in Section 4 that this is not the case.

We conjecture that this can be strengthened to the following:

Assume $\langle \mathscr{S}_k: k<\omega \rangle$ is a sequence of sets of finite graphs such that for all $k<\omega$ there is an H with $\mathrm{Chr}(H) \geqq \aleph_1$ and $\mathscr{S}(H) \subset \mathscr{S}_k$. Then there exists a graph G with $\mathrm{Chr}(G) \geqq \aleph_1$, and such that

$$\mathscr{S}_k \not\subset \mathscr{S}(G) \quad \text{for all} \quad k < \omega.$$

1. The positive result

Theorem 1. *If* $\mathrm{Chr}(G) \geqq \aleph_1$, *then G contains a subgraph isomorphic to* Γ.

Proof. Let $\varkappa = |G|$. The proof goes by induction on $\varkappa$. If $\varkappa \leqq \aleph_0$ the statement is vacuously true. Assume that $\varkappa \geqq \aleph_1$ and the statement is already proved for every smaller cardinality.

Put $V(G) = \{t_\alpha: \alpha < \varkappa\}$. By a simple Löwenheim—Skolem type argument we can build an increasing sequence of subsets V_α of V with the following properties:

(a) $t_\alpha \in V_{\alpha+1}$,

(b) $|V_\alpha| < \varkappa$,

(c) $V_\alpha = \bigcup \{V_\beta: \beta < \alpha\}$ if α is limit,

(d) if $\{x_0, \ldots, x_{s-1}\} \subset V_\alpha$ and there is a $y \in V \setminus V_\alpha$ joined to each of the $x_0, \ldots, x_{s-1}$, then there is such a y in $V_{\alpha+1} \setminus V_\alpha$ as well, for all $s<\omega$.

Put $W_\alpha = V_{\alpha\omega+\omega} \setminus V_{\alpha\omega}$. The sets $\{W_\alpha: \alpha<\varkappa\}$ give a partition of V into sets of smaller cardinality. By the induction hypothesis, the subgraphs induced by these sets have chromatic number $\leqq \aleph_0$. If for every $\beta<\varkappa$, every point in W_β is connected to finitely many points of $\bigcup \{W_\gamma: \gamma<\beta\}$ only, a well-known (see e.g. [3]) and easy recoloring process gives a good coloring of G with countably many colors.

Assume that $a \in W_\beta$ is joined to $x_0, x_1, \ldots$ with $\{x_i: i<\omega\} \subset \bigcup \{W_\gamma: \gamma<\beta\}$. For every $i<\omega$, $\{x_0, \ldots, x_{i-1}\} \subset V_{\beta\omega}$. By (d) of the construction there is a point

$y_i \in V_{\beta\omega}$, joined to each of the $x_0, \ldots, x_{i-1}$; $y_i \neq y_j$ for $j<i$. To make sure that $\{x_0, x_1, \ldots\} \cap \{y_0, y_1, \ldots\} = \emptyset$ take two disjoint subsets of these sets. The vertices $\{x_i, y_i, a\colon i<\omega\}$ show that Γ is isomorphic to a subgraph of G. ▮

Note that Theorem 1 yields the following.

(*) *For every* $G=\langle V, E\rangle$ *with* $\mathrm{Chr}(G) \geqq \aleph_1$ *there exists a* $k_0<\omega$ *and an edge* $e \in E$, *such that* e *is contained in a circuit of length* k *of* G *for all* $k_0 \leqq k < \omega$.

Indeed, in [4] the following lemma is proved.

Lemma. *Assume* $\mathrm{Chr}(G) \geqq \aleph_1$, *and*
$E' = \{e \in E\colon$ *there is even path of* G *connecting the endpoints of* $e\}$
$G' = \langle V, E'\rangle$.
Then $\mathrm{Chr}(G') \geqq \aleph_1$. ▮

Applying Theorem 1 for G' we get (*) easily.
(*) was proved earlier by Thomassen [10].

2. Construction of a large chromatic graph

In this section we modify a construction of [7] to give a subgraph G of chromatic number $\aleph_1$ of the comparability graph of a partially ordered set and such that G does not contain circuits which are the union of two increasing paths.

In the following construction $\{T_\alpha\colon \alpha<\omega_1\}$ will be a sequence of $\aleph_1$ disjoint sets, each of cardinality $\aleph_1$. Put $T = \bigcup\{T_\alpha\colon \alpha<\omega_1\}$. We shall construct a graph G with vertex set T by successively defining for every $x \in T_\alpha$ the set $G(x) = \{y \in \bigcup_{\beta<\alpha} T_\beta\colon y$ and x are joined$\}$. The sets T_α will be independent in G. G has the *H—M property* (see [6]) if for every x either $G(x)$ is finite or $\{\beta<\alpha\colon T_\beta \cap G(x) \neq \emptyset\}$ is of order type ω and cofinal in α. G is *special* if there is no circuit of the form $C=\{x_0, x_1, \ldots, x_{s-1}\}$ with $x_i \in T_{\alpha_i}$ and $\alpha_0<\alpha_1<\ldots<\alpha_t>\alpha_{t+1}>\ldots>\alpha_{s-1}>\alpha_0$ (note that G, if special, does not contain a triangle).

Theorem 2. (CH). *There exists a graph* G *on* T *with the following properties:*

(a) $\mathrm{Chr}(G) = \aleph_1$,
(b) G *is special,*
(c) G *has the H—M property,*
(c) *for* $x, y \in T$, $x \neq y$, $G(x) \cap G(y)$ *is finite.*

Proof. Let $\omega_1 = \bigcup\{X_\xi\colon \xi<\omega_1\}$ be a decomposition of ω_1 into $\aleph_1$ disjoint stationary sets.

We define $G(x)$ by induction on the levels. Assume that G has already been defined on $\bigcup\{T_\beta\colon \beta<\alpha\}$ and α is limit.

Whenever $\gamma<\alpha$ and $y \in \bigcup\{T_\beta\colon \beta<\alpha\}$, y is said to be γ-covered if there exists a monotone path of G, $\{x_0, \ldots, x_s\}$, $x_s = y$ with $x_i \in T_{\alpha_i}$ and $\gamma \geqq \alpha_0 < \ldots < \alpha_s$.

Assume that $\alpha \in X_\xi$ where $\xi<\alpha$ (otherwise put $G(x)=\emptyset$ for $x \in T_\alpha$). Let us define $\mathscr{W}_\alpha$, the set of α-candidates as the set of pairs $\langle W, f\rangle$ satisfying the following conditions:

a) W is a countable subset of $\cup\{T_\beta: \beta<\alpha\}$,
b) $W=\{x_\tau: \tau<\omega^2\}$ and $x_\tau\in T_{\alpha_\tau}$ where $\alpha_\tau<\alpha_{\tau'}$ for $\tau<\tau'<\omega^2$,
c) $\sup\{\alpha_\tau: \tau<\omega^2\}=\alpha$,
d) $\xi<\alpha_0$,
e) no x_τ is $\sup\{\alpha_{\tau'}: \tau'<\tau\}$-covered for $\tau<\omega^2$.
f) $f: W\to\omega$, and there is an $n<\omega$ satisfying the following condition:

$$(+)\qquad f^{-1}(\{n\})\cap\{x_\tau: \tau\in[\omega i, \omega(i+1))\} \text{ is infinite for all } i<\omega.$$

Clearly, $|\mathscr{W}_\alpha|\leq\aleph_1$. If $\mathscr{W}_\alpha=\emptyset$ put $G(x)=\emptyset$ for $x\in T_\alpha$. So we may assume that there is an enumeration $\mathscr{W}_\alpha=\{\langle W_\eta, f_\eta\rangle: \eta<\omega_1\}$. Our intention is to choose a $G(z_\eta)\subset W_\eta$ for every $\eta<\omega_1$ for the elements of $T_\alpha=\{z_\eta: \eta<\omega_1\}$ in such a way that the sets $\{G(z_\eta): \eta<\omega_1\}$ are almost disjoint and satisfy the H—M property.

Assume that $\eta<\omega_1$ and $\{G(z_{\eta'}): \eta'<\eta\}$ are defined. Put $W=W_\eta$, $f=f_\eta$. Enumerate as $\{n_0, n_1, \ldots\}$ those n's which satisfy (+). For $i<\omega$ choose y_i from

$$f^{-1}(\{n_i\})\cap\{x_\tau: \tau\in[\omega i, \omega(i+1))\}\setminus\bigcup_{j<i} G(z_{h(j)})$$

where $\{h(j): j<\omega\}=\eta$ is a reordering of η into a sequence of type ω. Put $G(z_\eta)=\{y_i: i<\omega\}$.

We have to check the properties (a)—(d) for the graph G. (c) and (d) are clear (let us note that CH is used only to satisfy (d)). For (b) let us assume that $C=\{x_0, \ldots, x_{s-1}\}$, $s<\omega$ is a circuit with $x_i\in T_{\alpha_i}$,

$$\alpha_0<\alpha_1<\ldots<\alpha_t>\alpha_{t+1}>\ldots>\alpha_{s-1}>\alpha_0.$$

Assume for the sake of definiteness that $\alpha_{t-1}<\alpha_{t+1}$. Then $x_{t-1}, x_{t+1}\in G(x_t)$ and x_{t+1} is α_{t-1}-covered, a contradiction.

To check (a), assume that $f: T\to\omega$ is a good coloring of G. A color $n<\omega$ is called *small* if there is a $\gamma_n<\omega_1$ such that every point $x\in T$ with $f(x)=n$ is γ_n-covered. Otherwise, call n large. Put $\gamma=\sup\{\gamma_n: n \text{ small}\}$. If n is large, the set

$$C_n=\{\alpha<\omega_1: \text{ if } \gamma<\alpha \text{ there is an } x\in\cup\{T_\beta: \beta<\alpha\} \text{ with } f(x)=n \text{ and } x \text{ is not } \gamma\text{-covered}\}$$

is clearly closed unbounded. Now choose an element α with $\alpha>\gamma$, $\alpha\in X_\gamma$ and there is a monotone sequence $\{c_\tau: \tau<\omega^2\}\subset\cap\{C_n: n<\omega\}$ cofinal in α, $c_0\in T_{\alpha_0}$, $\alpha_0>\gamma$. By induction we can choose $\{x_\tau: \tau<\omega^2\}$ such that x_0 is not γ-covered, $x_\tau\in T_{\alpha_\tau}$, $\alpha_\tau\in[c_\tau, c_{\tau+1})$; no x_τ is $\sup\{\alpha_{\tau'}: \tau'<\tau\}$-covered and, if n is large, $f(x_{\omega i+n})=n$. As the pair $\langle W, f|W\rangle$ with $W=\{x_\tau: \tau<\omega^2\}$ is appropriate for a)—f) of the construction, there is a $y\in T_\alpha$, $G(y)\subset W$, $G(y)\cap f^{-1}(\{n\})\neq\emptyset$ for every large n. We show that y cannot have a color. Sure, $f(y)$ is small, so y is γ-covered, but this is impossible, as $G(y)\subset W$ and no element of W is γ-covered. This contradiction shows that $\mathrm{Chr}(G)=\aleph_1$. ∎

3. A graph without Δ

Theorem 3. *The graph in Theorem 2 does not contain a subgraph isomorphic to Δ.*

Proof. Assume that $\{x_i, y_i, a, b: i<\omega\}\subset T$. We can assume that $x_i\in T_{\alpha_i}$, $\alpha_0<\alpha_1<\dots$. As $\{x_0, x_1, \dots\}\not\subset G(a)\cap G(b)$, one or both of a, b, say a, has level less than $\sup\{\alpha_i: i<\omega\}$. If $a\in T_\beta$, $\beta<\alpha_i$ and $y_j\in T_{\gamma_j}$, then $\gamma_j<\alpha_{i+1}$ for $j>i+1$, otherwise $\{a, x_i, y_j, x_{i+1}\}$ would form a forbidden circuit. But then $\{y_{i+k}: 2\leq k<\omega\}\subset \subset G(x_{i+1})\cap G(x_{i+2})$, a contradiction. ▮

4. The answer to the E-H-S problem

Theorem 4. *There is a graph $G=\langle V, E\rangle$, and a partial order $\prec$ on V, such that, $|V|=2^{\aleph_0}$, $\mathrm{Chr}(G)=\aleph_1$; $\{x, y\}\in E\Rightarrow x\prec y\vee y\prec x$ and G is special with respect to the partial order.* ▮

The proof of this is very similar to the proof of Theorem 3. One has to choose the levels T_α to be of size $2^{\aleph_0}$ for $\alpha<\omega_1$. We omit the details.

We now claim that the graph G satisfying the requirements of Theorem 4 does not contain $\mathcal{S}(G_k(\omega))$ for $2\leq k<\omega$. Assume that $2\leq k<\omega$ and that for $r<\omega$ $r\to(2k+1)_2^{k+1}$ holds, and assume indirectly that f: $V(G_k(r))\to V$ is an embedding of $G_k(r)$ into V. Then there is an $A\subset r$, $|A|=2k+1$ such that either $f(\{x_0, \dots, x_{k-1}\})\prec \prec f(\{x_1, \dots, x_k\})$ holds for all $\{x_0, \dots, x_k\}_<\in[A]^{k+1}$ or $f(\{x_0, \dots, x_k\})\succ f(\{x_1, \dots, x_k\})$ holds for all $\{x_0, \dots, x_k\}_<\in[A]^{k+1}$.

It is a matter of trivial computation to see that $f''[A]^k$ contains a circuit of G special for $\prec$ in both cases.

Added in proof. The proof of Theorem 1 gives also that every graph with uncountable *coloring* number (see [3]) contains Γ. The second author has found a more complicated countable graph K contained in every graph of uncountable coloring number which is also universal with respect to this property, i. e. K contains every countable graph sharing this property. A similar example of size $\aleph_1$ has also been found. These results will soon be published.

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